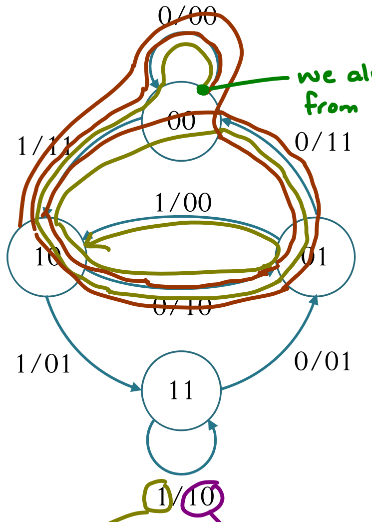
ECS 452: Digital Communication Systems **2015/2**

HW 7 — Due: Not Due

Lecturer: Asst. Prof. Dr. Prapun Suksompong

Problem 1. Consider a convolutional encoder whose state diagram is given in Figure 7.1.



we always start from the 00 state.

Figure 7.1: State diagram for a convolutional encoder

(a) Find the code rate

one bit $\Rightarrow k=1$

two bits $\Rightarrow n=2$

code rate $= \frac{k}{n} = \frac{1}{2}$

(b) Suppose the data bits (message) are 0100101. Find the corresponding codeword.

From the path on the state diagram, we have $\underline{x}$ = [00111011111000]

(c) Find the data vector $\underline{\mathbf{b}}$ which gives the codeword $\underline{\mathbf{x}}$ = [0011101111110110011]

From the path on the state diagram, we have $\underline{b}$ = [010010001]

Problem 2. Consider a convolutional encoder whose trellis diagram is given in Figure 7.2.

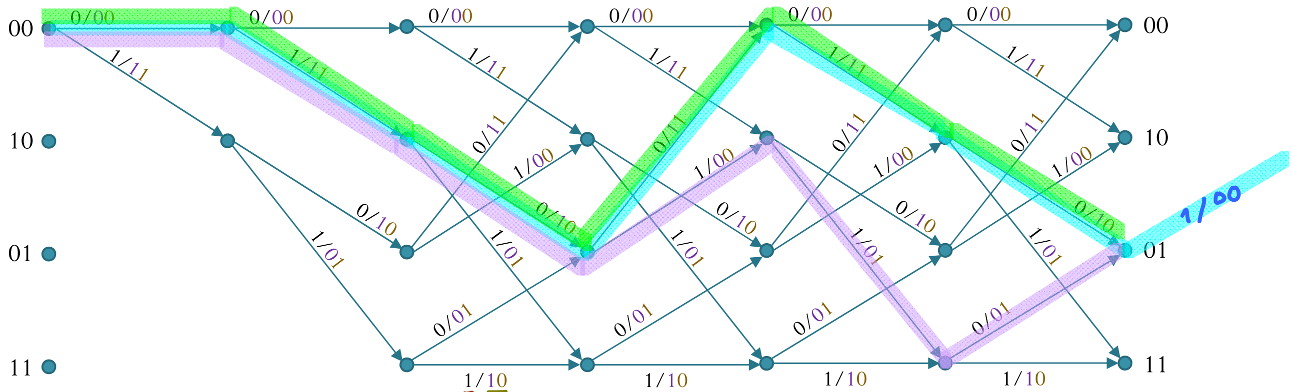


Figure 7.2: State diagram for a convolutional encoder

(a) Find the code rate $\frac{\text{one bit}}{\text{two bits}} = \frac{1}{2}$

(b) Suppose the data bits (message) are $\underline{\mathbf{b}} = [0100101]$. Find the corresponding codeword $\underline{\mathbf{x}}$.

From the blue highligthed path, we have $\underline{x}$ = [00111011111000]

(c) Find the data vector $\underline{\mathbf{b}}$ which gives the codeword $\underline{\mathbf{x}} = [001110111110]$.

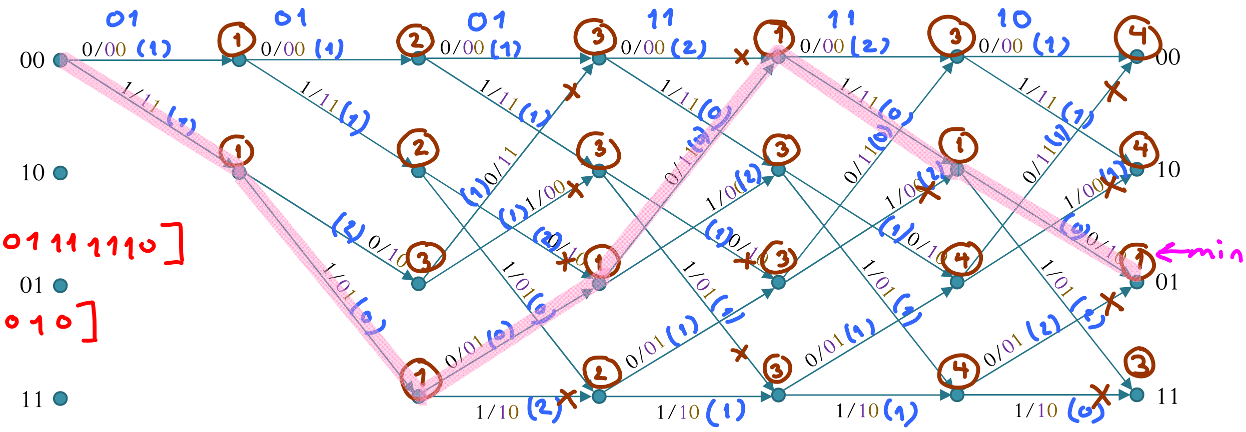
From the green highlighted path, we have $\underline{b}$ = [010010]

Alternatively, because the codeword here is the same as the first 12 bits of the codeword in the previous part, we know that the data vector must be the same as the first 6 bits of the data vector from the previous part.

(d) Suppose that we observe $\underline{\mathbf{y}} = [001110000101]$ at the input of the minimum distance decoder. Explain why we can easily find the decoded codeword $\hat{\underline{\mathbf{x}}}$ and the decoded message $\hat{\underline{\mathbf{b}}}$ without applying the Viterbi algorithm.

From the purple highligthed path, we see that $\underline{y}$ itself is a codeword.
So, there is a codeword ($\underline{y}$) with distance = 0 from $\underline{y}$.
There can not be any codeword with smaller distance from $\underline{y}$ than 0.
So, $\hat{\underline{x}} = \underline{y}$.
From the purple highligthed path, we can also read $\hat{\underline{b}}$ = [010110]

(e) Suppose that we observe $\underline{\mathbf{y}} = [010101111110]$ at the input of the minimum distance decoder. Use Viterbi algorithm to find the decoded codeword $\hat{\underline{\mathbf{x}}}$ and the decoded message $\hat{\underline{\mathbf{b}}}$. Show your work on Figure 7.3 below.



$\hat{\underline{x}} = [110101111110]$

$\hat{\underline{b}} = [110010]$

Figure 7.3: State diagram for a convolutional encoder

Make sure that all the running (cumulative) path metric are shown and the discarded branches are indicated at every steps.

Problem 3. Consider the two signals $s_1(t)$ and $s_2(t)$ shown in Figure 7.4. Note that V and T_b are some positive constants. Your answers should be given in terms of them.

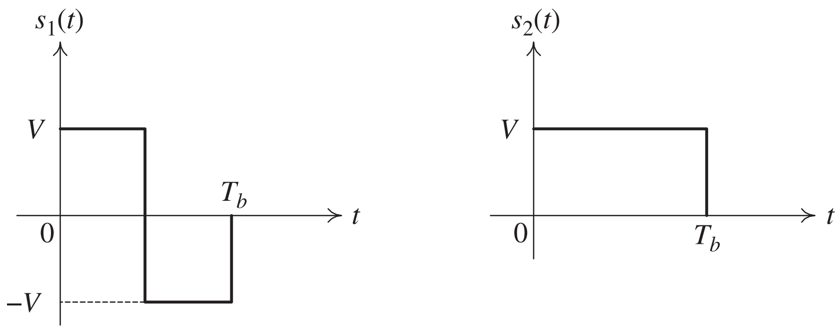


Figure 7.4: Signal set for Question 3

(a) Find the energy in each signal.

$$E_1 = E_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

$$E_2 = E_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

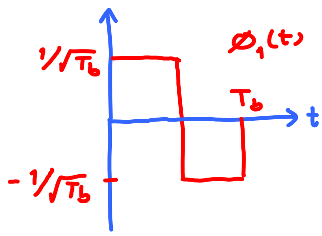
$$\equiv E_s$$

(b) Use the Gram-Schmidt orthogonalization procedure (GSOP) (where the signals are applied **in the order given**) to find two orthonormal functions $\phi_1(t)$ and $\phi_2(t)$ that can be used to represent $s_1(t)$ and $s_2(t)$.

$u_1(t) = s_1(t)$.

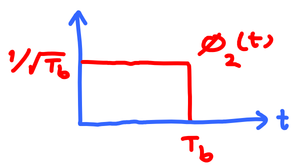
$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{1}{\sqrt{E_s}} s_1(t) = \begin{cases} 1/\sqrt{T_b}, & 0 \le t \le \frac{T_b}{2} \\ -1/\sqrt{T_b}, & \frac{T_b}{2} < t < T_b \\ 0, & \text{otherwise} \end{cases}$$

$$\frac{V}{\sqrt{E_s}} = \frac{V}{V\sqrt{T_b}} = \frac{1}{\sqrt{T_b}}$$



$$u_2(t) = s_2(t) - \text{proj}_{u_1} s_2 = s_2(t) - \frac{\overbrace{\langle s_2, s_1\rangle}^{0}}{\langle s_1, s_1\rangle} s_1(t) = s_2(t)$$

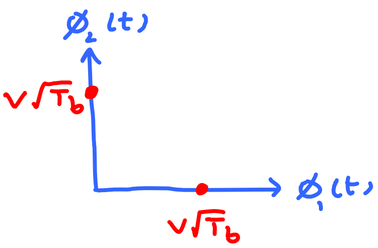
$$\phi_2(t) = \frac{u_2(t)}{\sqrt{E_{u_2}}} \underset{u_2 = s_2}{=} \frac{s_2(t)}{\sqrt{E_2}} = \frac{1}{\sqrt{E_s}} s_2(t) = \begin{cases} 1/\sqrt{T_b}, & 0 \le t \le T_b, \\ 0, & \text{otherwise.} \end{cases}$$



(c) Find the two vectors that represent the two waveforms $s_1(t)$ and $s_2(t)$ in the new (signal) space based on the orthonormal basis found in the previous part. Draw the corresponding constellation.

$$s_1(t) = \sqrt{E_s}\,\phi_1(t) + 0\,\phi_2(t) \Rightarrow \vec{s}^{(1)} = \begin{pmatrix} \sqrt{E_s} \\ 0 \end{pmatrix} = \begin{pmatrix} V\sqrt{T_b} \\ 0 \end{pmatrix}.$$

$$s_2(t) = 0\,\phi_1(t) + \sqrt{E_s}\,\phi_2(t) \Rightarrow \vec{s}^{(2)} = \begin{pmatrix} 0 \\ \sqrt{E_s} \end{pmatrix} = \begin{pmatrix} 0 \\ V\sqrt{T_b} \end{pmatrix}.$$



Problem 4. Consider the two signals $s_1(t)$ and $s_2(t)$ shown in Figure 7.5. Note that V, α and T_b are some positive constants.

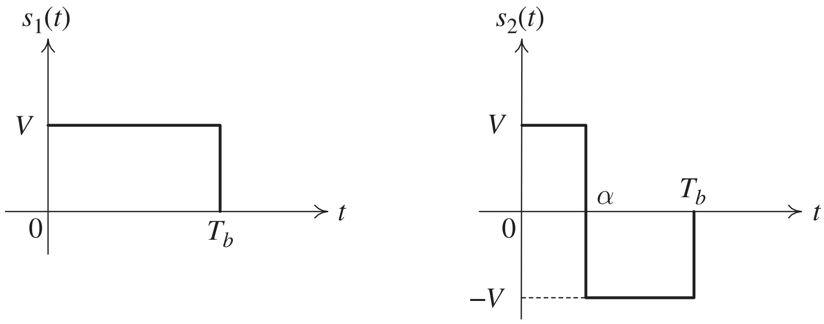


Figure 7.5: Signal set for Question 4

(a) Find the energy in each signal.

$$E_1 = E_{s_1} = \int_{-\infty}^{\infty} |s_1(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

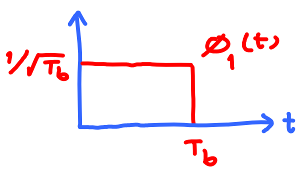
$$E_2 = E_{s_2} = \int_{-\infty}^{\infty} |s_2(t)|^2 dt = \int_0^{T_b} V^2 dt = V^2 T_b.$$

$$\equiv E_s$$

(b) Use the Gram-Schmidt orthogonalization procedure (GSOP) (where the signals are applied **in the order given**) to find two orthonormal functions $\phi_1(t)$ and $\phi_2(t)$ that can be used to represent $s_1(t)$ and $s_2(t)$.

$u_1(t) = s_1(t).$

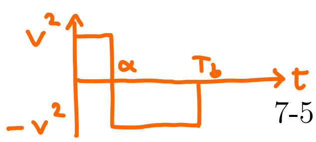
$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{1}{\sqrt{E_s}} s_1(t) = \begin{cases} 1/\sqrt{T_b}, & 0<t<T_b \\ 0, & \text{otherwise.} \end{cases}$$



$$\frac{V}{\sqrt{E_s}} = \frac{V}{V\sqrt{T_b}} = \frac{1}{\sqrt{T_b}}$$

$$u_2(t) = s_2(t) - \text{proj}_{u_1} s_2 = s_2(t) - \frac{\langle s_2, s_1\rangle}{\langle s_1, s_1\rangle} s_1(t)$$

$\langle s_2, s_1\rangle = \int$

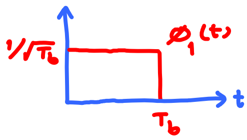


dt $= \alpha V^2 - (T_b - \alpha) V^2 = 2\alpha V^2 - T_b V^2$

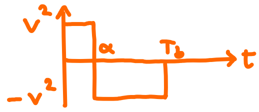
$u_1(t) = s_1(t)$.

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \frac{1}{\sqrt{E_1}} s_1(t) = \begin{cases} 1/\sqrt{T_b}, & 0<t<T_b \\ 0, & \text{otherwise.} \end{cases}$$

$$\frac{V}{\sqrt{E_a}} = \frac{V}{V\sqrt{T_b}} = \frac{1}{\sqrt{T_b}}$$



$$u_2(t) = s_2(t) - \text{proj}_{u_1} s_2 = s_2(t) - \frac{\langle s_2, s_1 \rangle}{\langle s_1, s_1 \rangle} s_1(t)$$

$\langle s_2, s_1 \rangle = \int$  dt

$$= \alpha V^2 - (T_b - \alpha)V^2 = 2\alpha V^2 - T_b V^2$$

$$u_2(t) = s_2(t) - \frac{2\alpha \cancel{V^2} - T_b \cancel{V^2}}{\cancel{V^2} T_b} s_1(t) = s_2(t) - \left(\frac{2\alpha}{T_b} - 1\right) s_1(t)$$

$$= s_2(t) + \underbrace{\left(1 - \frac{2\alpha}{T_b}\right)} s_1(t)$$

When $\alpha = \frac{T_b}{2}$, this term $= 0$ and $u_2(t) = s_2(t)$.

When $\alpha < \frac{T_b}{2}$, this term is > 0. So, $s_2(t)$ will be shifted up.

When $\alpha > \frac{T_b}{2}$, this term is < 0. So, $s_2(t)$ will be shifted down.

$$= \begin{cases} V + \left(1 - \frac{2\alpha}{T_b}\right)V & \text{for } 0<t<\alpha, \\ -V + \left(1 - \frac{2\alpha}{T_b}\right)V & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} \left(2 - \frac{2\alpha}{T_b}\right)V & \text{for } 0<t<\alpha, \\ -\frac{2\alpha}{T_b}V & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases}$$

$$= 2V \times \begin{cases} 1 - \frac{\alpha}{T_b} & \text{for } 0<t<\alpha, \\ -\frac{\alpha}{T_b} & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases}$$

Note: The shift amount is just enough to make $\int u_2 = 0$.

$$\left[\left(1 - \frac{\alpha}{T_b}\right)\alpha + \left(-\frac{\alpha}{T_b}\right)(T_b - \alpha) = \alpha - \frac{\alpha^2}{T_b} - \alpha + \frac{\alpha^2}{T_b} = 0.\right]$$

$$\mathcal{E}_{u_2} = 4V^2\left(\underbrace{\left(1 - \frac{\alpha}{T_b}\right)^2}_{1 - \frac{2\alpha}{T_b} + \frac{\alpha^2}{T_b^2}}\alpha + \left(-\frac{\alpha}{T_b}\right)^2 (T_b - \alpha)\right) = 4V^2\left(\alpha - \frac{\cancel{2}\alpha^2}{T_b} + \cancel{\frac{\alpha^3}{T_b^2}} + \cancel{\frac{\alpha^2}{T_b}} - \cancel{\frac{\alpha^3}{T_b^2}}\right)$$

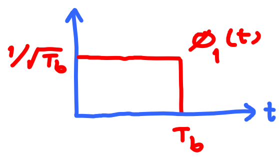
$$= 4V^2\alpha\left(1 - \frac{\alpha}{T_b}\right) \quad \Rightarrow \sqrt{\mathcal{E}_{u_2}} = 2V\sqrt{\alpha\left(1 - \frac{\alpha}{T_b}\right)}$$

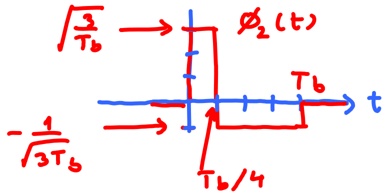
$$\phi_2(t) = \frac{u_2(t)}{\sqrt{\mathcal{E}_{u_2}}} = \frac{2\cancel{V}}{\cancel{2V}\sqrt{\alpha\left(1 - \frac{\alpha}{T_b}\right)}} \begin{cases} 1 - \frac{\alpha}{T_b} & \text{for } 0<t<\alpha, \\ -\frac{\alpha}{T_b} & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \frac{1}{\sqrt{\alpha\left(1 - \frac{\alpha}{T_b}\right)}} \times \begin{cases} 1 - \frac{\alpha}{T_b} & \text{for } 0<t<\alpha, \\ -\frac{\alpha}{T_b} & \text{for } \alpha<t<T_b, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Plot $\phi_1(t)$ and $\phi_2(t)$ when $\alpha = \frac{T_b}{4}$. $\Rightarrow \frac{1}{\sqrt{\alpha(1-\frac{\alpha}{T_b})}} = \frac{1}{\sqrt{\frac{T_b}{4}(1-\frac{1}{4})}} = \frac{4}{\sqrt{3T_b}}$

$$\phi_2(t) = \frac{4}{\sqrt{3T_b}} \times \begin{cases} 3/4 & \text{for } 0<t<\frac{T_b}{4} \\ -1/4 & \text{for } \frac{T_b}{4}<t<T_b \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \sqrt{3/T_b}, & \text{for } 0<t<\frac{T_b}{4}, \\ -\frac{1}{\sqrt{3T_b}}, & \text{for } \frac{T_b}{4}<t<T_b, \\ 0, & \text{otherwise.} \end{cases}$$





(d) Find the two vectors $\boldsymbol{s}^{(1)}$ and $\boldsymbol{s}^{(2)}$ that represent the two waveforms $s_1(t)$ and $s_2(t)$ in the new (signal) space based on the orthonormal basis found in the previous part.

$$s_1(t) = \sqrt{E_1}\,\phi_1(t) = \sqrt{E_s}\,\phi_1(t) = v\sqrt{T_b}\,\phi_1(t) \Rightarrow \vec{s}^{(1)} = \begin{pmatrix} v\sqrt{T_b} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} \\ 0 \end{pmatrix}.$$

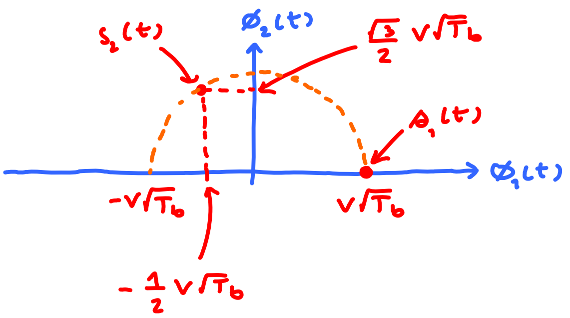
$$s_2(t) = u_2(t) - \left(1 - 2\frac{\alpha}{T_b}\right)s_1 = \sqrt{\mathcal{E}_{u_2}}\,\phi_2(t) - \left(1 - 2\frac{\alpha}{T_b}\right)\sqrt{\mathcal{E}}\,\phi_1(t)$$

$$= 2v\sqrt{\alpha\left(1-\frac{\alpha}{T_b}\right)}\,\phi_2(t) - \left(1-2\frac{\alpha}{T_b}\right)v\sqrt{T_b}\,\phi_1(t).$$

$$\Rightarrow \vec{s}^{(2)} = \begin{pmatrix} -\left(1-2\frac{\alpha}{T_b}\right) \\ 2\sqrt{\frac{\alpha}{T_b}\left(1-\frac{\alpha}{T_b}\right)} \end{pmatrix} \times v\sqrt{T_b} = \sqrt{E_s}\begin{pmatrix} 2r-1 \\ 2\sqrt{r(1-r)} \end{pmatrix} \quad \text{where } r = \frac{\alpha}{T_b}.$$

(e) Draw the corresponding constellation when $\alpha = \frac{T_b}{4}$.

When $\alpha = \frac{T_b}{4}$, $s_2(t) = 2v\sqrt{\frac{T_b}{4}\left(\frac{3}{4}\right)}\,\phi_2(t) - \left(1-\frac{1}{2}\right)v\sqrt{T_b}\,\phi_1(t) = \frac{\sqrt{3}}{2}v\sqrt{T_b}\,\phi_2(t) - \frac{1}{2}v\sqrt{T_b}\,\phi_1(t)$



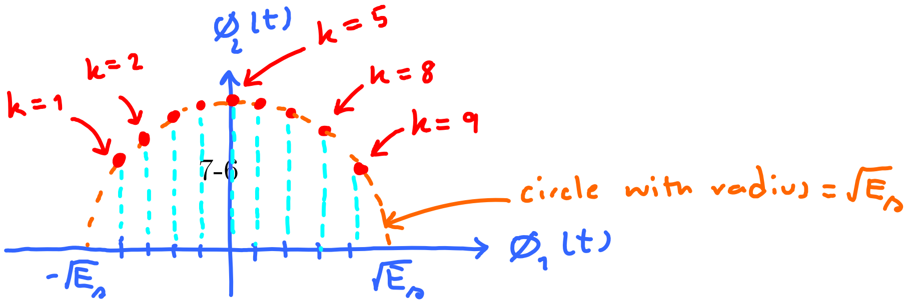
(f) Draw $\boldsymbol{s}^{(2)}$ when $\alpha = \frac{k}{10}T_b$ where $k = 1, 2, \ldots, 9$.

Note that

$(2r-1)^2 + \left(2\sqrt{r(1-r)}\right)^2$

$= 4r^2 - 4r + 1 + 4r - 4r^2 = 1.$

$r = \frac{\alpha}{T_b} = \frac{k}{10}$



Problem 5. In a ternary signaling scheme, the message S is randomly selected from the alphabet set $\mathcal{S} = \{-1, 1, 4\}$ with $p_1 = P[S = -1] = 0.41$, $p_2 = P[S = 1] = 0.08$ and $p_3 = P[S = 4] = 0.51$. Find the average signal energy E_s.

Recall that $E_s = \sum_{j=1}^{M} p_j E_j$ where E_j = energy of $s_j(t)$ = $\langle s_j(t), s_j(t) \rangle$ = energy of $\vec{s}^{(j)}$ = $\langle \vec{s}^{(j)}, \vec{s}^{(j)} \rangle = \sum_{i=1}^{K} |\underbrace{s_i^{(j)}}|^2$

the i^{th} element in vector $\vec{s}^{(j)}$

There are three possible $\vec{s}$

↳ $\vec{s}^{(1)}, \vec{s}^{(2)}, \vec{s}^{(3)}$

Here, $M = 3$ and $K = 1$, $\vec{s}$ is one-dimensional (a scalar)

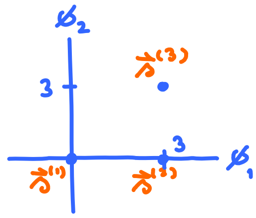
So, $E_j = |s^{(j)}|^2$ and $E_s = \sum_{j=1}^{3} p_j |s^{(j)}|^2$

$= 0.41 \times (-1)^2 + 0.08 \times (1)^2 + 0.51 \times 4^2 = 8.65$

Problem 6. Consider a ternary constellation. Assume that the three vectors are equiprobable.

(a) Suppose the three vectors are

$$\boldsymbol{s}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{s}^{(2)} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \text{ and } \boldsymbol{s}^{(3)} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$



Find the corresponding average energy per symbol.

$E_s = \frac{1}{3}\left(0^2 + 3^2 + 3^2 + 3^2\right) = 9$

(b) Suppose we can shift the above constellation to other location; that is, suppose that the three vectors in the constellation are

$$\boldsymbol{s}^{(1)} = \begin{pmatrix} 0 - a_1 \\ 0 - a_2 \end{pmatrix}, \boldsymbol{s}^{(2)} = \begin{pmatrix} 3 - a_1 \\ 0 - a_2 \end{pmatrix}, \text{ and } \boldsymbol{s}^{(3)} = \begin{pmatrix} 3 - a_1 \\ 3 - a_2 \end{pmatrix}.$$

Find a_1 and a_2 such that corresponding average energy per symbol is minimum.

\# dimensions. Here, $K = 2$.

$E_s = \sum_{j=1}^{M} \underbrace{p_j}_{\text{probability of } \vec{s}^{(j)}} \underbrace{E_j}_{\text{energy of } \vec{s}^{(j)}} = \sum_{j=1}^{M} p_j \sum_{i=1}^{K} |\underbrace{\vec{s}_i^{(j)}}_{\text{the } i^{th} \text{ element in the vector } \vec{s}^{(j)}}|^2 = \sum_{i=1}^{2} \sum_{j=1}^{M} p_j |\vec{s}_i^{(j)}|^2$

$= \frac{1}{3}\left((0-a_1)^2 + (2-a_1)^2 + (2-a_1)^2\right) + \frac{1}{3}\left((0-a_2)^2 + (0-a_2)^2 + (2-a_2^2)\right)$

Calculus: See next page

Minimum E_s occurs when

$a_1 = \frac{1}{3}(0+3+3) = 2$. $a_2 = \frac{1}{3}(0+0+3) = 1$.

The first elements in the vectors. The second elements in the vectors.

Remark for Q6

In general, we have to minimize terms of the form $\sum_{j=1}^{M} p_j (x_j - a)^2$.

Method 1:

$$\sum_{j=1}^{M} p_j (x_j - a)^2 = \mathbb{E}[(X-a)^2] = \mathbb{E}[(X - \underbrace{\mathbb{E}X + \mathbb{E}X}_{0} - a)^2]$$

$$= \operatorname{Var} X + 2\,\underbrace{\mathbb{E}[(X - \mathbb{E}X)]}_{0}(\mathbb{E}X - a) + (\mathbb{E}X - a)^2$$

$$= \operatorname{Var} X + (\underbrace{\mathbb{E}X - a}_{})^2$$

This is the only term that depends on a. Minimum value of 0 is achieved when $a = \mathbb{E}X$.

Method 2:

$$\sum_{j=1}^{M} p_j (x_j - a)^2 = \sum_{j=1}^{M} p_j (x_j^2 - 2a x_j + a^2) = a^2 - 2a\mathbb{E}X + \mathbb{E}[X^2]$$

As a function of "a", this is a parabola with minimum at $a = \mathbb{E}X$.

set $\frac{d}{da}(\ \ ) = 0$. Then solve for a.

Method 3: We can try to find the derivative right from the beginning.

$$\frac{d}{da} \sum_{j=1}^{M} p_j (x_j - a)^2 = -\sum_{j=1}^{M} p_j\, 2(x_j - a) = -2(\mathbb{E}X - a)$$

The derivative above is 0 when $a = \mathbb{E}X$.

when $a < \mathbb{E}X$, the slope is <0

when $a > \mathbb{E}X$, the slope is >0

So, the minimum value occurs when $a = \mathbb{E}X = \sum_i p_i x_i$

Problem 7. Prove the following facts with the help of Fourier transform. (Hint: inner product in the frequency domain, Parseval's theorem)

(a) The energy of $p(t) = g(t)\cos(2\pi f_c t + \phi)$ is $E_g/2$.

(b) $g(t)\cos(2\pi f_c t)$ and $-g(t)\sin(2\pi f_c t)$ are orthogonal.

Is there any condition(s) on $g(t)$ for this technique to work?

(a)

$$p(t) = g(t)\cos(2\pi f_c t + \phi) = \frac{g(t)}{2}\left(e^{j(2\pi f_c t+\phi)} + e^{-j(2\pi f_c t+\phi)}\right)$$

$\cos A = \frac{e^{jA}+e^{-jA}}{2}$

$$= \frac{g(t)}{2}e^{j\phi}e^{j2\pi f_c t} + \frac{g(t)}{2}e^{-j\phi}e^{-j2\pi f_c t}$$

$\downarrow \mathcal{F}$

$$P(f) = \frac{e^{j\phi}}{2}G(f-f_c) + \frac{e^{-j\phi}}{2}G(f+f_c)$$

$$E_p = \langle p(t), p(t)\rangle = \langle P(f), P(f)\rangle = \int_{-\infty}^{\infty} P(f)P^*(f)\,df$$

$$= \int_{-\infty}^{\infty}\Big(\underbrace{\frac{e^{j\phi}}{2}G(f-f_c)}_{A} + \underbrace{\frac{e^{-j\phi}}{2}G(f+f_c)}_{B}\Big)\Big(\underbrace{\frac{e^{-j\phi}}{2}G^*(f-f_c)}_{C} + \underbrace{\frac{e^{j\phi}}{2}G^*(f+f_c)}_{D}\Big)df$$

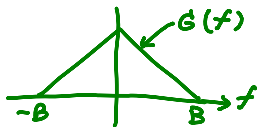
$$= \int_{-\infty}^{\infty}(A+B)(C+D)\,df = \int_{-\infty}^{\infty}AC\,df + \int_{-\infty}^{\infty}AD\,df + \int_{-\infty}^{\infty}BC\,df + \int_{-\infty}^{\infty}BD\,df$$

Now, note that

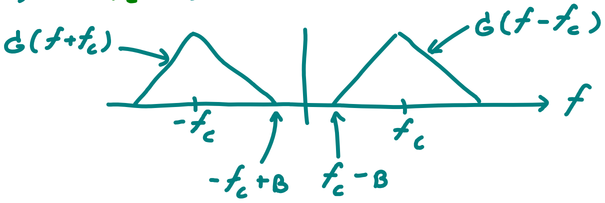
$$AC = \frac{1}{4}|G(f-f_c)|^2$$
$$BD = \frac{1}{4}|G(f+f_c)|^2$$

suppose we assume that $G(f)$ is band-limited to $\pm B$, i.e.,

$$G(f) = 0 \quad \text{when } |f| > B.$$



Then, if $f_c > B$,



the non-zero parts of $G(f-f_c)$ and $G(f+f_c)$ do not overlap.

Therefore, $\left.\begin{array}{l} BC \equiv 0 \\ AD \equiv 0 \end{array}\right\}$ across all f.

So, $E_p = \int_{-\infty}^{\infty} \frac{1}{4}|G(f-f_c)|^2 df + 0 + 0 + \int_{-\infty}^{\infty} \frac{1}{4}|G(f+f_c)|^2 df$

$\mu = f - f_c$ $\qquad$ $\mu = f + f_c$ $\qquad$ (change of variables)

$= \int_{-\infty}^{\infty} \frac{1}{4}|G(\mu)|^2 d\mu + \int_{-\infty}^{\infty} \frac{1}{4}|G(\mu)|^2 d\mu$

$= \frac{1}{4}E_g + \frac{1}{4}E_g = \frac{E_g}{2}$.

This formula is valid when $G(f) = 0$ for $|f| \geq f_c$.

(b)

$$x(t) = g(t)\cos(2\pi f_c t) \xrightarrow{\mathcal{F}} X(f) = \frac{1}{2}\Big(\underbrace{G(f-f_c)}_{A} + \underbrace{G(f+f_c)}_{B}\Big)$$

$$y(t) = -g(t)\underbrace{\sin(2\pi f_c t)}_{\frac{1}{2j}\left(e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right)} \xrightarrow{\mathcal{F}} Y^*(f) = +\frac{1}{2j}\Big(\underbrace{G^*(f-f_c)}_{C} - \underbrace{G^*(f+f_c)}_{D}\Big)$$

$$\langle x, y\rangle = \langle X, Y\rangle = \int_{-\infty}^{\infty} X(f)Y^*(f)\,df = \frac{1}{4j}\int_{-\infty}^{\infty} \underbrace{(A+B)(C-D)}_{=AC-AD+BC-BD}\,df$$

$$= \frac{1}{4j}\int_{-\infty}^{\infty} |G(f-\cancel{f_c})|^2 df - \frac{1}{4j}\int_{-\infty}^{\infty} |G(f+\cancel{f_c})|^2 df + 0 + 0$$

$= 0$

$G(f-f_c)$ and $G(f+f_c)$ do not overlap.

Assumption: $G(f)$ is bandlimited

$G(f) = 0$ for $f > f_c$, $f < -f_c$